

Nadir Pointing with Coupled Spacecraft Attitude and Fuel Slosh Dynamics using Hybrid Robust Adaptive Control

AE 544 Analytical Dynamics
Final Project

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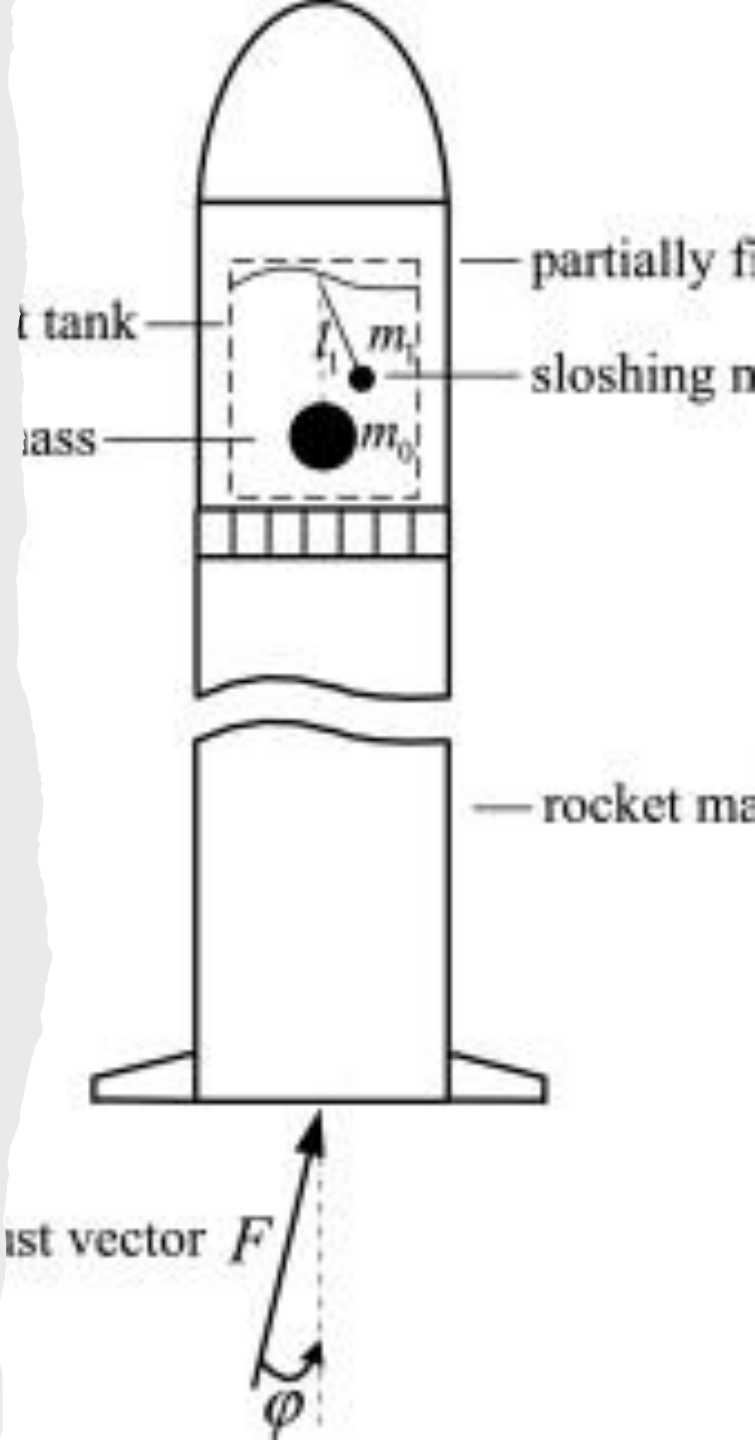
Aerospace Engineering

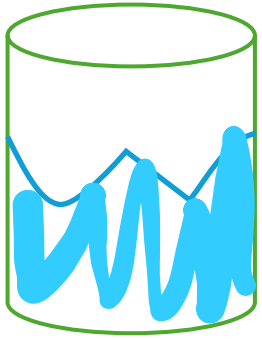


Introduction/ Problem Description



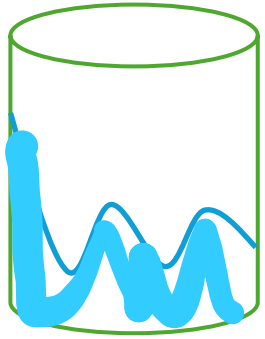
Early Mission (Full Tank)

- Large fuel volume → **large slosh mass**
- Low slosh natural frequency (liquid moves more slowly)
- Strong coupling with spacecraft attitude dynamics
- Disturbance torques are significant



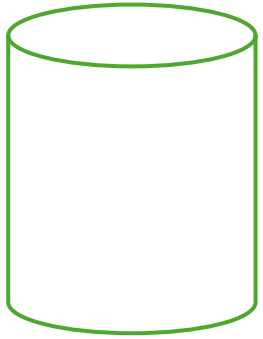
Mid Mission (Partially Filled Tank)

- Slosh mass decreases as fuel is used
- Slosh frequency increases (a shorter “pendulum length,” depending on tank geometry)
- Disturbance torque reduces but still influences pointing
- Coupling becomes more nonlinear as tank geometry interacts with motion



Late Mission (Nearly Empty Tank)

- Very little propellant left
- Slosh mode has **small amplitude but high frequency**
- Effects diminish but do **not** disappear until the tank is nearly empty
- Residual slosh can still create small, repetitive disturbances



End of Life (Tank Depletion)

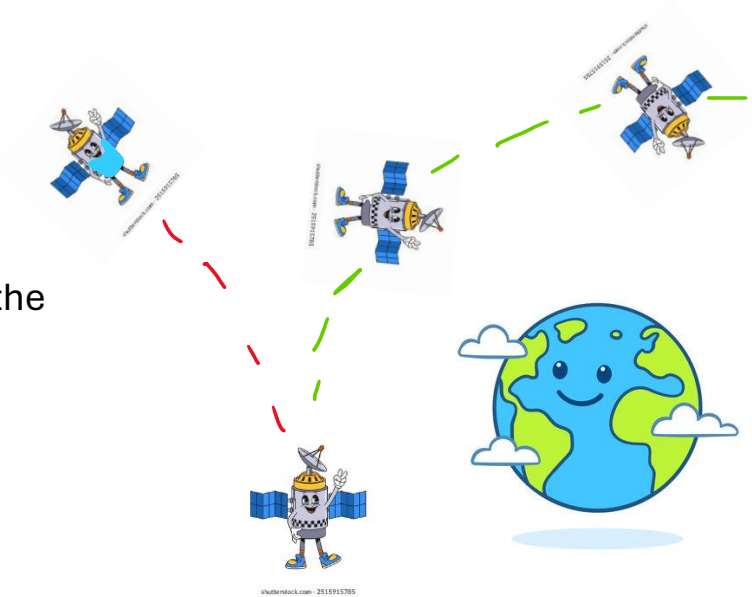
- No slosh once the liquid is gone
- Spacecraft behaves as a rigid body
- Attitude controller no longer needs to compensate for moving mass

Introduction/ Problem Description

The goal of this project is to ****maintain stable nadir pointing**** of a spacecraft under ****time-varying fuel slosh dynamics****, using a ****reaction wheel as the primary actuator****, while incorporating a ****thruster-based backup mechanism**** to handle actuator saturation or failure.

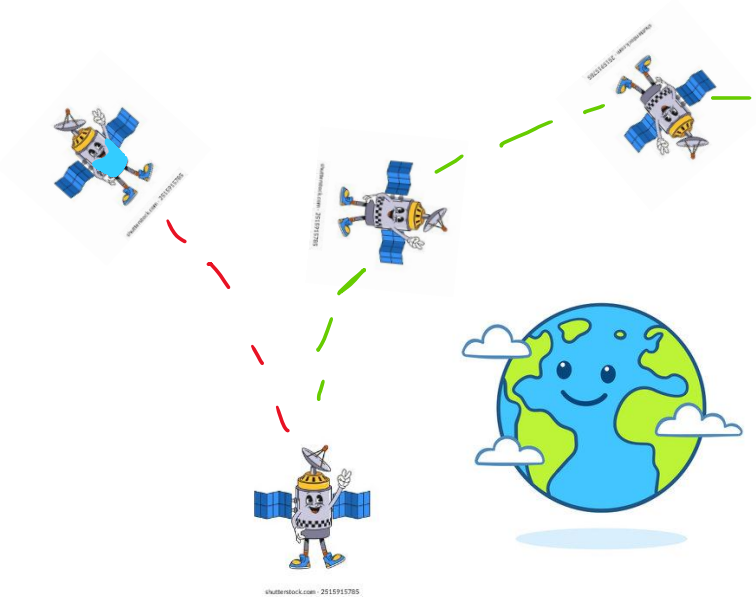
Assumptions:

- Spacecraft attitude as a **single-axis rotation**
- Fuel slosh as a **pendulum-like internal mass**
- Coupling between slosh motion and spacecraft rotation
- External torque control inputs (e.g., reaction wheels or thrusters)
- Fuel is being consumed over time
- After fuel is gone, decouple the dynamics and active controller effect
- The reaction wheel and thruster are modeled as **ideal torque sources** and therefore enter the system as a **single combined control** input rather than dynamic states. This simplifies the model while preserving the essential coupling between actuation and spacecraft–slosh dynamics.



Core challenge

- Non-linear time-varying dynamics (NLTV system)
- Implementing a robust adaptive controller



Introduction/ Problem Description

4 DOF model:

🛰️ **1. Attitude angle dynamics**

- θ
- $\dot{\theta}$

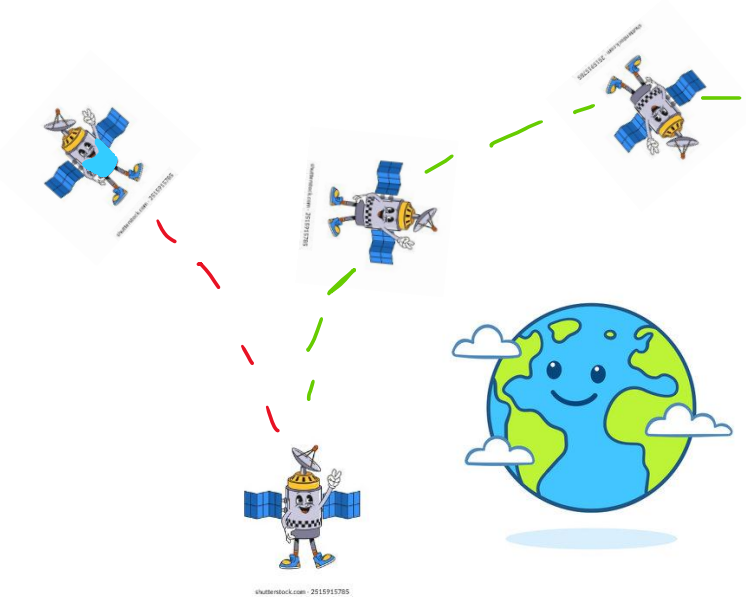
👉 This is **1 rotational DOF (rigid body attitude)**

🌊 **2. Slosh dynamics**

- ϕ
- $\dot{\phi}$

👉 This is **1 internal oscillatory DOF (fluid motion)**

2 generalized coordinates → 4 state-space DOF



Derivation (Nadir Pointing)

Orbital angular velocity

For a circular orbit:

$$\omega_{orbit} = \sqrt{\frac{\mu}{r^3}}$$

Where,

- $\mu = GM$ (gravitational parameter)
- r = orbital radius

Orbital velocity relationships

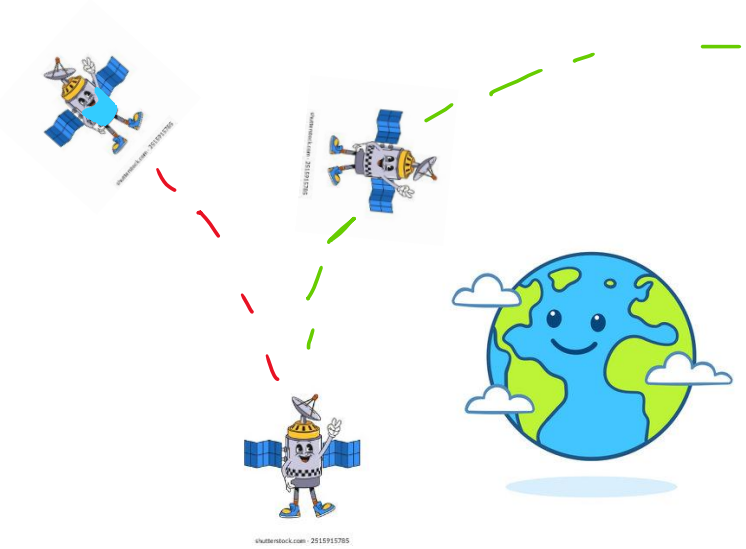
Starting from the centripetal balance:

$$\frac{\mu m}{r^2} = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{\mu}{r}} \quad <--- \text{velocity}$$

$$v = r\omega$$

$$\omega = \frac{v}{r} = \sqrt{\frac{\mu}{r^3}} \quad <------ \text{angular velocity}$$



[2] Vallado, D. A., Analytical Mechanics of Space Systems, 2nd ed., American Institute of Aeronautics and Astronautics, Reston, VA, 2013.

Derivation (Nadir Pointing)

Nadir-pointing reference angle

$$\theta_d(t) = \omega_{orbit} t$$

This assumes:

- constant orbital rate
- inertial frame reference starting at $t = 0$

So equivalently:

$$\theta_d(t) = \sqrt{\frac{\mu}{r^3}}\, t$$

Gravity gradient disturbance torque

$$\tau_{gg} = \frac{3\mu}{r^3}\left(I_{dry} - I_3\right)\sin(2\theta)$$

is consistent if:

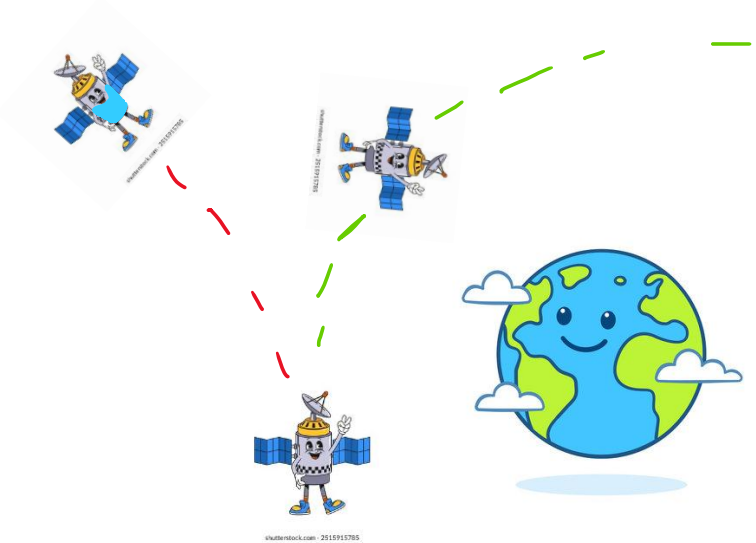
- I_3 is the principal moment aligned with nadir
- I_{dry} is the transverse principal moment

Key idea:

- torque comes from inertia asymmetry + Earth gravity gradient
- it drives passive alignment of long axis toward nadir

Gravity at Earth surface

$$g_{eff} = 9.81\ m/s^2$$



Derivation (Actuators Architecture)

The reaction wheel equation

$$\tau_{rw} = -I_w \dot{\omega}_w$$

captures angular momentum exchange correctly, but it implicitly assumes:
•perfect torque transmission (no friction, no motor dynamics),
•instantaneous wheel acceleration capability,

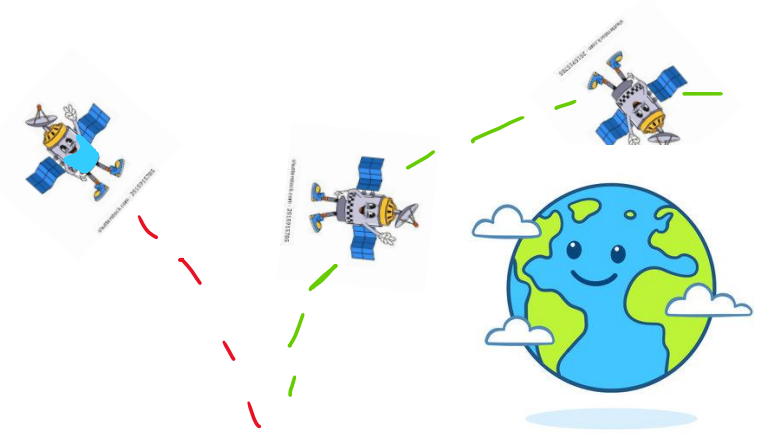
For the thruster model

$$\tau_{thr} = Fd$$

a thruster produces torque via external force.

Total Control Torque

$$\tau_{ext} = u = u_{rw} + u_{thr}$$



Derivation (Actuators Architecture)

Reaction wheel (Primary)
Applied torque (saturated)

$$u_{rw} = sat\left(u_{cmd},\ u_{rw,\max}\right)$$
$$|\ u_{rw}\ | \leq u_{rw,max} = 0.5\ N$$

Thruster (Backup)
Residual torque

$$u_{thr} = u_{cmd} - u_{rw}$$

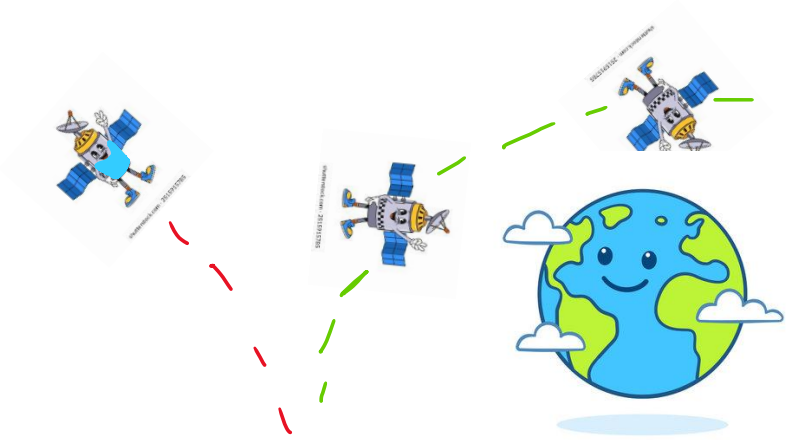
Applied torque (saturated)

$$u_{thr} = sat\left(u_{thr},\ u_{thr,\max}\right)$$
$$|\ u_{thr}\ | \leq u_{thr,max} = 2.0\ N$$

Control Allocation Logic

- Reaction wheel supplies torque **first**
- Thruster supplies **only the leftover**

$$u_{thr} = \begin{cases} 0, & |\ u_{cmd}\ | \leq u_{rw,max} \\ u_{cmd} - u_{rw}, & otherwise \end{cases}$$



Derivation (Kinematics and Dynamics)

The inertial frame:

$$N = O, \hat{n}_1, \hat{n}_2, \hat{n}_3$$

Position of the spacecraft center of mass (SCM):

$$\mathbf{R}(t) = \overrightarrow{OO'}$$

The spacecraft body-fixed frame:

$$B = O', \hat{b}_1, \hat{b}_2, \hat{b}_3$$

A positive rotation θ rotates the inertial axes into the body axes through rotation about $\hat{n}_3$:

$$\hat{b}_1 = \cos\theta\,, \hat{n}_1 + \sin\theta\,, \hat{n}_2$$
$$\hat{b}_2 = -\sin\theta\,, \hat{n}_1 + \cos\theta\,, \hat{n}_2$$
$$\hat{b}_3 = \hat{n}_3$$

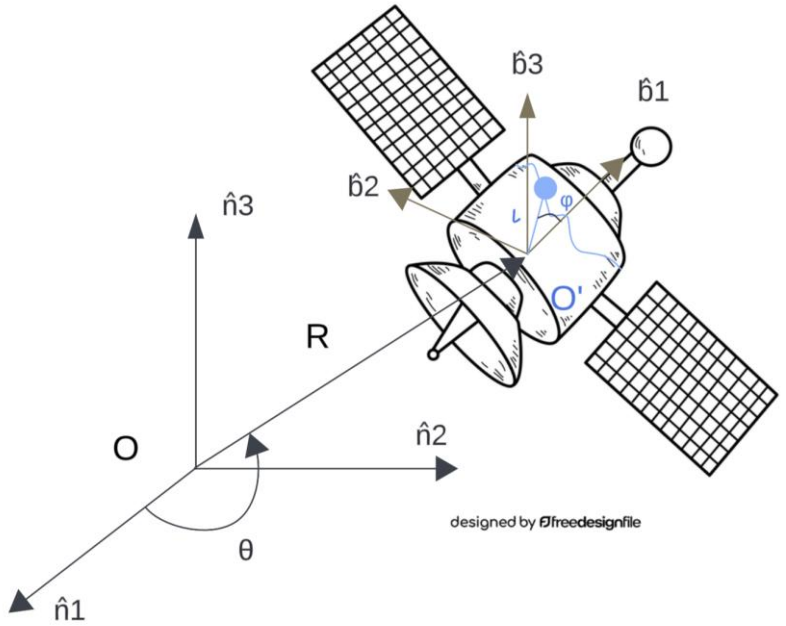
These satisfy:

$$\hat{b}_1 \times \hat{b}_2 = \hat{b}_3$$

The DCM maps inertial components into body components:

$$\mathbf{v}_B = C_{B\leftarrow N}, \mathbf{v}_N$$

$$C_{B\leftarrow N}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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Derivation (Kinematics and Dynamics)

Define the pendulum deflection angle ϕ:

- Measured from $+\hat{b}_1$
- Positive toward $+\hat{b}_2$

Pendulum tip in body coordinates:

$$\mathbf{l}_B(\phi) = \ell \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}$$

Pendulum vector in the inertial frame:
Using the DCM:

$$\mathbf{l}_N = C_{N\leftarrow B}(\theta), \mathbf{l}_B(\phi)$$

Total slosh mass in inertial frame (combine translation and rotation):

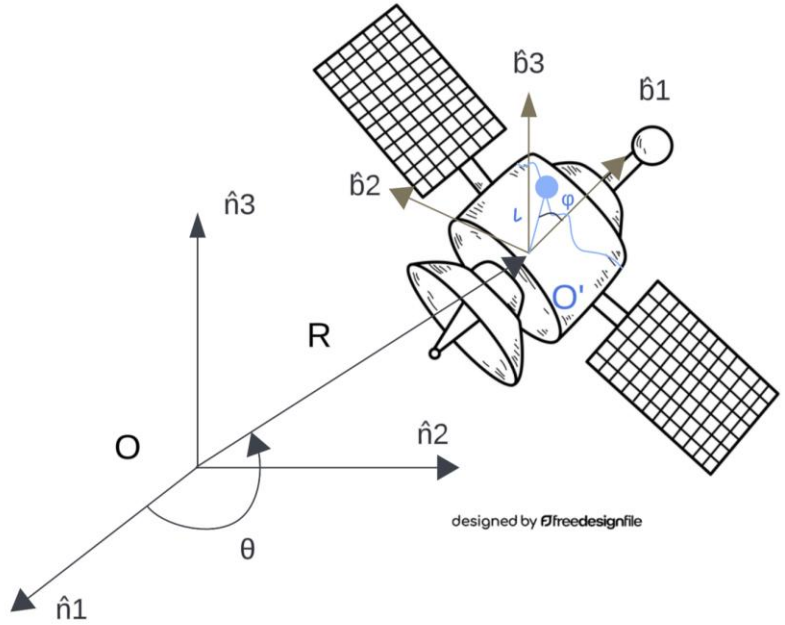
$$\mathbf{r}_s(t) = \mathbf{R}(t) + \ell \begin{bmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \\ 0 \end{bmatrix}$$

The velocity vector is:

$$\dot{\mathbf{r}}_s = \dot{\mathbf{R}} + \frac{d\ell}{dt}\begin{pmatrix} \cos(\theta+\phi) \\ sin(\theta+\phi) \\ 0 \end{pmatrix} + \ell(\dot{\theta}+\dot{\phi})\begin{pmatrix} -\sin(\theta+\phi) \\ cos(\theta+\phi) \\ 0 \end{pmatrix}$$

Radial velocity

Tangential velocity



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Derivation (Kinematics and Dynamics)

Lagrangian Formulation:
The velocity squared is:

$$|\dot{\mathbf{r}}_s|^2 = \dot{\mathbf{r}}_s \cdot \dot{\mathbf{r}}_s$$

$$\boxed{|\dot{\mathbf{r}}_s|^2 = \left|\dot{\mathbf{R}}\right|^2 + \dot{\ell}^2 + \ell^2\left(\dot{\theta} + \dot{\phi}\right)^2 + 2\dot{\mathbf{R}} \cdot \left(\dot{\ell}\hat{\mathbf{e}}_r + \ell\left(\dot{\theta} + \dot{\phi}\right)\hat{\mathbf{e}}_\perp\right)}$$

The spacecraft rotates about its symmetry axis.

Rotational inertia about b_3:

$$I_3(t) = I_{dry} + m_s(t)\ell^2.$$

I_{dry} is the dry mass inertia, and $m_s(t)\ell^2$ is the contribution from the slosh mass.

Thus:

$$T_{sc} = \frac{1}{2} I_3(t)\dot{\theta}^2.$$

Choose a COM frame where $\dot{\mathbf{R}} = \mathbf{0}$:

$$|\dot{\mathbf{r}}_s|^2 = \dot{\ell}^2 + \ell^2\left(\dot{\theta} + \dot{\phi}\right)^2.$$

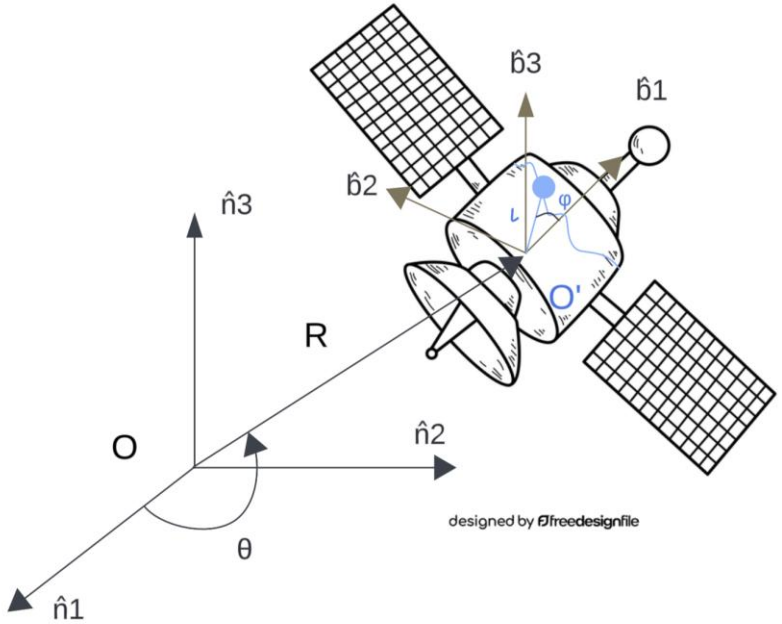
Slosh kinetic energy:

$$T_s = \frac{1}{2} m_s(t)\left[\dot{\ell}^2 + \ell^2\left(\dot{\theta} + \dot{\phi}\right)^2\right].$$

The pendulum deflects by angle ϕ under an effective gravity field g_{eff}.

Zero potential at $\phi = 0$:

$$V = m_s(t) g_{eff} \ell(1 - \cos\phi).$$



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Derivation (Kinematics and Dynamics)

$$\mathcal{L} = T - V = \frac{1}{2} I_3(t)\dot{\theta}^2 + \frac{1}{2} m_s(t)\left[\dot{\ell}^2 + \ell^2\left(\dot{\theta} + \dot{\phi}\right)^2\right] - m_s(t) g_{eff} \ell (1 - \cos\phi)$$

Lagrange equation for θ (Attitude Equation)

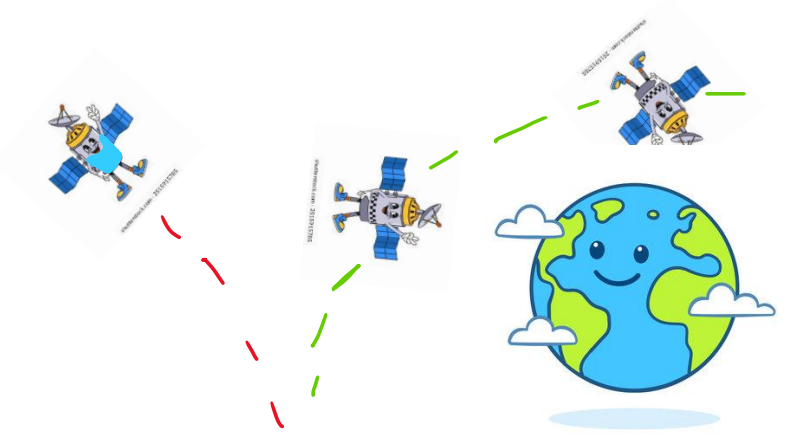
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \tau_{ext}$$

$$I_3\ddot{\theta} + m_s\ell^2\ddot{\phi} + \ell^2\dot{m}_s\left(2\dot{\theta} + \dot{\phi}\right) + 2m_s\ell\dot{\ell}\left(\dot{\theta} + \dot{\phi}\right) = \tau_{ext}$$

Lagrange equation for ϕ (Slosh Equation)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

$$\ddot{\theta} + \ddot{\phi} + \frac{\dot{m}_s}{m_s}\left(\dot{\theta} + \dot{\phi}\right) + 2\frac{\dot{\ell}}{\ell}\left(\dot{\theta} + \dot{\phi}\right) + \frac{g_{eff}}{\ell}\sin\phi = 0$$



Derivation (State Space Modeling)

Define the state vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

Control input:

$$u = \tau_{ext}$$

$$f_\theta(x, t) = \frac{1}{I_{dry}}\left[-\ell^2 \dot{m}_s (x_2 + x_4) + m_s \ell g_{eff} \sin(x_3)\right]$$

$$f_\phi(x, t) = \frac{1}{I_{dry}}\left[(x_2 + x_4)\left(2\ell^2 \dot{m}_s - I_3 \frac{\dot{m}_s}{m_s} + 2m_s \ell \dot{\ell} - 2I_3 \frac{\dot{\ell}}{\ell}\right) - \frac{I_3 g_{eff}}{\ell} \sin(x_3)\right]$$

$$\dot{x}_1 = x_2$$

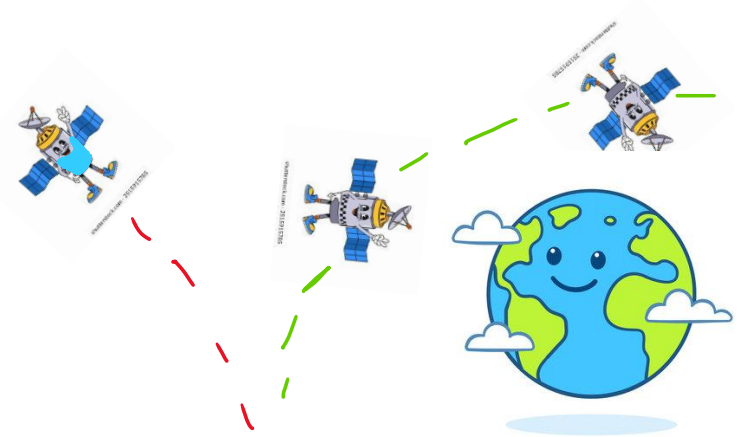
$$\dot{x}_2 = f_\theta(x, t) + \frac{1}{I_{dry}}(u_{rw} + u_{thr})$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = f_\phi(x, t) - \frac{1}{I_{dry}}(u_{rw} + u_{thr})$$

Therefore, the state space model is:

$$\dot{x} = \begin{bmatrix} x_2 \\ f_\theta(x, t) + I_{dry}^{-1}(u_{rw} + u_{thr}) \\ x_4 \\ f_\phi(x, t) - I_{dry}^{-1}(u_{rw} + u_{thr}) \end{bmatrix}$$

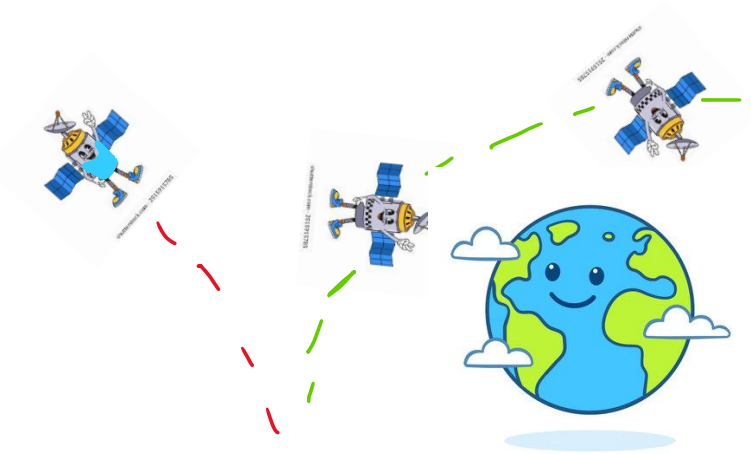


Derivation (Linearization) – Why needed?

👉 What kind of stability analysis are you doing?
Lyapunov Indirect Method. So, we will study the behavior around the **linearized dynamical system** and extrapolate the conclusion to the broader non-linear system.

👉 Why cannot you study the non-linear system to prove Lyapunov Stability?
This system does NOT satisfy Linear Parameterization. You can only do Lyapunov Stability analysis and design an MRAC (Model Adaptive Reference Control) controller directly for a nonlinear system if it is linearly parameterizable.

👉Why is this system not linearly parametrizable?
Because of non-linear terms, $\sin x_3$, and the time varying parameters.



Derivation (Linearization)

Linearization about (x*, u*):

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{x*,u*,t}, B(t) = \left.\frac{\partial f}{\partial u}\right|_{x*,u*,t}$$

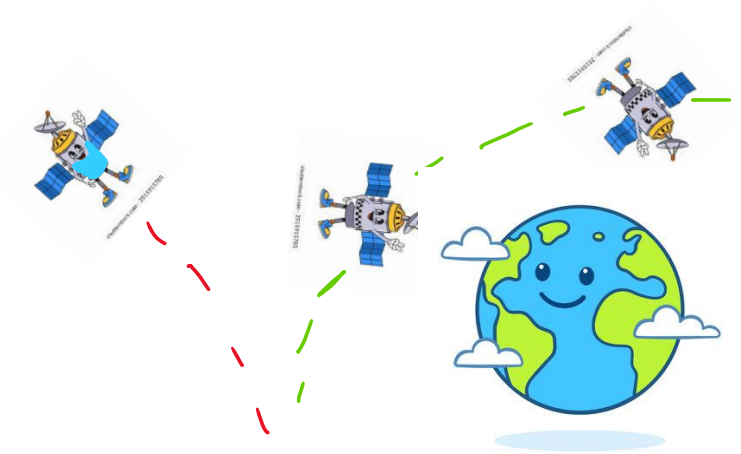
The Jacobian of the nonlinear system linearized about

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u^* = 0$$

is

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\ell^2 \dot{m}_s}{I_{dry}} & \frac{m_s \ell g_{eff}}{I_{dry}} & -\frac{\ell^2 \dot{m}_s}{I_{dry}} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{I_{dry}}\left(2\ell^2 \dot{m}_s - I_3 \frac{\dot{m}_s}{m_s} + 2m_s \ell \dot{\ell} - 2I_3 \frac{\dot{\ell}}{\ell}\right) & -\frac{I_3}{I_{dry}} \frac{g_{eff}}{\ell} & \frac{1}{I_{dry}}\left(2\ell^2 \dot{m}_s - I_3 \frac{\dot{m}_s}{m_s} + 2m_s \ell \dot{\ell} - 2I_3 \frac{\dot{\ell}}{\ell}\right) \end{bmatrix}$$

$$B(t) = \begin{bmatrix} 0 \\ \frac{1}{I_{dry}} \\ 0 \\ -\frac{1}{I_{dry}} \end{bmatrix}$$



Derivation (Frequency and Damping Ratio)

MRAC Reference Model

```
Am_coupled = [0 1 0 0;
             -2 -2 0 0;
              0 0 0 1;
              0 0 -2 -2];
```

and

```
Am_rigid = [0 1 0 0;
           -3 -2 0 0;
            0 0 0 1;
            0 0 -3 -2];
```

Each 2×2 block is a **standard second-order system** of the form:

$$\dot{x}_1 = x_2, \dot{x}_2 = -ax_1 - bx_2$$

which corresponds to:

$$\ddot{x} + b\dot{x} + ax = 0$$

So we match:

- Natural frequency: $\omega_n = \sqrt{a}$
- Damping ratio: $\zeta = \frac{b}{2\sqrt{a}}$

•Coupled model

From:

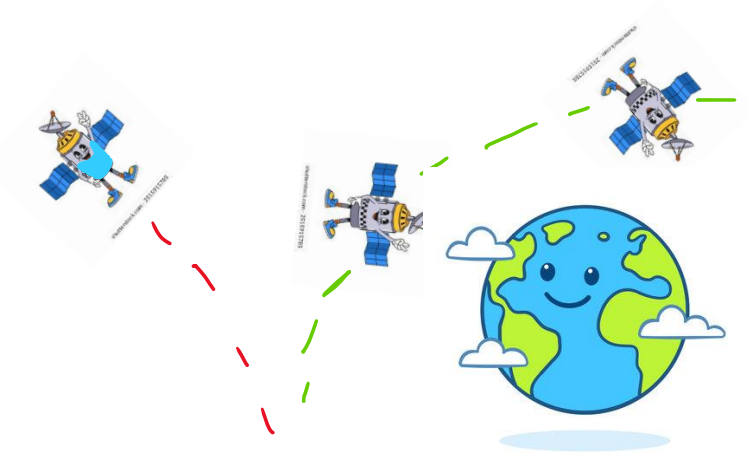
$$a = 2, b = 2$$

Natural frequency:

$$\omega_n = \sqrt{2} \approx 1.414\ rad/s$$

Damping ratio:

$$\zeta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \approx 0.707$$ **well-damped (near critical damping)**



Derivation (Frequency and Damping Ratio)

Rigid model
From:

$$a = 3, b = 2$$

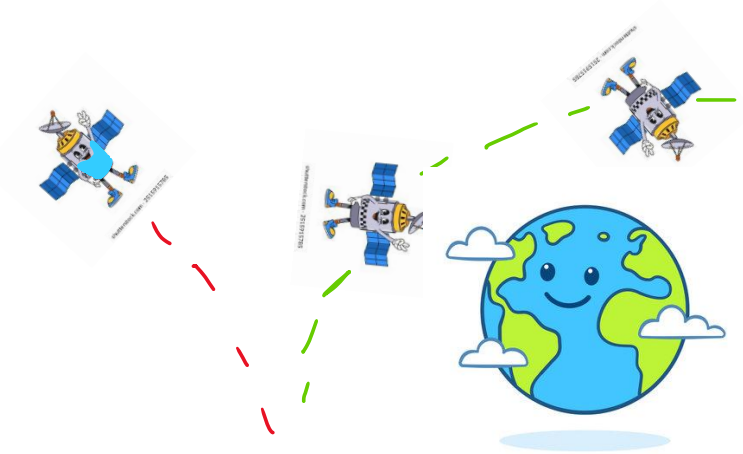
Natural frequency:

$$\omega_n = \sqrt{3} \approx 1.732 \; rad/s$$

Damping ratio:

$$\zeta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.577$$ **moderately underdamped**

Important note
The system is 4th-order, but structured as **two decoupled 2nd-order subsystems**:
•attitude channel
•slosh/channel dynamics



Derivation (Control and Stability)

Nonlinear Time-Varying Plant Model
The spacecraft dynamics are given by:

$$\dot{x}(t) = A(x, t)x(t) + Bu(t) + Bd(t)$$

State vector:

$$x = \left[\theta, \dot{\theta}, \phi, \dot{\phi}\right]^T$$

Disturbance model:

$$d(t) = \tau_{gg} + \Delta_{sat} + \Delta_{model}$$

Δ_{sat} *is actuator saturation loss. Can be injected to increase fidelity.*
Δ_{model} *is modeling mismatch. Can be injected to increase fidelity.*

Gravity-gradient torque:

$$\tau_{gg} = \frac{3\mu}{r^3}\left(I_{dry} - I_3\right)\sin(2\theta)$$

Hybrid Reference Model
Reference dynamics:

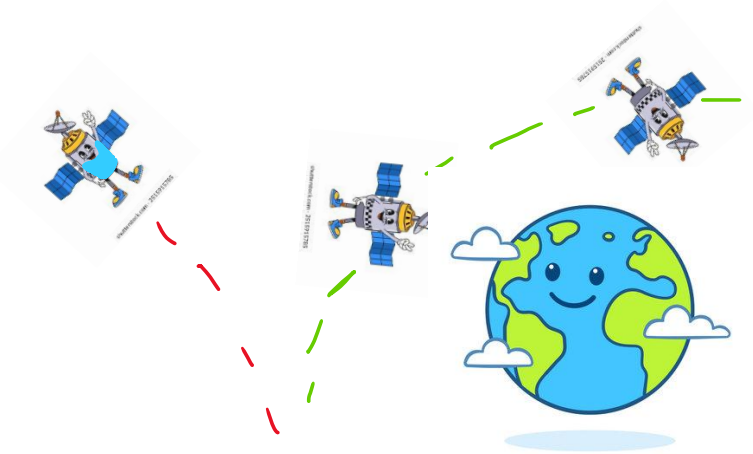
$$\dot{x}_m = A_m(\alpha)x_m + B_m r$$

Gain scheduling:

$$A_m(\alpha) = \begin{cases} A_{m,c}, & \alpha > 0.05 \\ A_{m,r}, & \alpha \leq 0.05 \end{cases}$$

Input matrix:

$$B_m = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



Derivation (Control and Stability)

Tracking Error Dynamics
Tracking error:

$$e = x - x_m$$

Error dynamics:

$$\dot{e} = A_m e + B\widetilde{W}^T x + d(t)$$

Parameter estimation error:

$$\widetilde{W} = \widehat{W} - W^*$$

Hybrid Control Law
Control input:

$$u = -K(\alpha)x + \widehat{W}^T x + k_r(\alpha)r$$

Gain scheduling:

$$(K, k_r) = \begin{cases} (K_c, k_{r,c}), & \alpha > 0.05 \\ (K_r, k_{r,r}), & \alpha \leq 0.05 \end{cases}$$

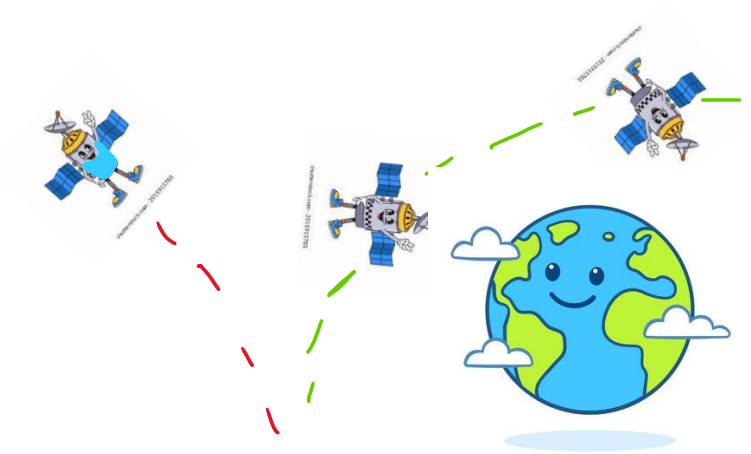
$$\alpha \; is \; fuel \; fraction \; remaining.$$

Adaptive Law (σ-Modification)
Adaptive update law:

$$\dot{\widehat{W}} = \begin{cases} -\Gamma x e^T P B_m - \sigma \widehat{W}, & \alpha > 0.05 \\ 0, & \alpha \leq 0.05 \end{cases}$$

Projection bounds:

$$\widehat{W} \in [-, W_{max} W_{max}]$$



Derivation (Control and Stability)

Actuator Allocation
Commanded control:

$$u_{cmd} = -Kx + \widehat{W}^T x + k_r r$$

Allocation:

$$u_{rw} = sat(u_{cmd}, u_{rw}^{max})$$
$$u_{thr} = sat(u_{cmd}, -u_{rw} u_{thr}^{max})$$

Final control:

$$u = u_{rw} + u_{thr}$$

Lyapunov Function
Candidate function:

$$V = e^T P e + \widetilde{W}^T \Gamma^{-1} \widetilde{W}$$

Lyapunov equation:

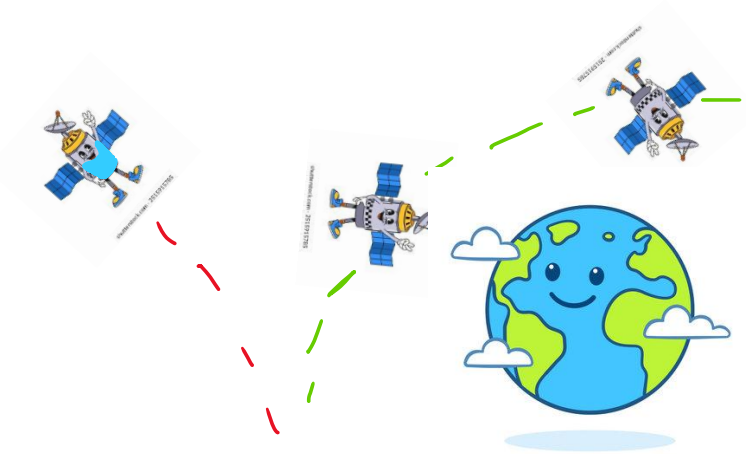
$$A_m^T P + P A_m = -Q, Q > 0$$

Stability Analysis
Time derivative:

$$\dot{V} = 2e^T P \dot{e} + 2\widetilde{W}^T \Gamma^{-1} \dot{\widetilde{W}}$$

Substitution:

$$\dot{V} = -e^T Q e + 2e^T P B \widetilde{W}^T x + 2e^T P d(t) - 2\sigma \widetilde{W}^T \Gamma^{-1} \widehat{W}$$



Derivation (Control and Stability)

Bounding Analysis

Using Young's inequality:

$$\dot{V} \leq -\lambda_{min}(Q) \parallel e \parallel^2 -c_1 \parallel \widetilde{W} \parallel^2 +c_2 \parallel e \parallel +c_3$$

Where:

$$c_1 = \sigma\lambda_{min}\left(\Gamma^{-1}\right) - \frac{4 \parallel PB \parallel^2 x_{max}^2}{\lambda_{min}(Q)}$$

$$c_2 = 2 \parallel P \parallel d_{max}$$

$$c_3 = \frac{2 \parallel P \parallel^2 d_{max}^2}{\lambda_{min}(Q)} + \frac{\sigma \parallel \Gamma^{-1} \parallel^2 \parallel W^* \parallel^2}{\lambda_{min}(\Gamma^{-1})}$$

Stability condition:

$$c_1 > 0$$

UUB Stability Result

Bounding Lyapunov dynamics:

$$\dot{V} \leq -\alpha V + \beta$$

Where:

$$\alpha = \min\left(\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}, \frac{c_1}{\lambda_{max}(\Gamma^{-1})}\right)$$

$$\beta = \frac{c_2^2}{4\lambda_{min}(Q)} + c_3$$

Conclusion:

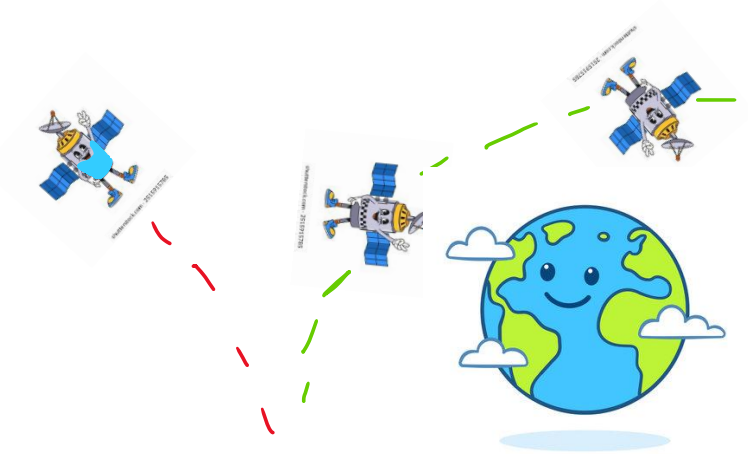
- System is **Uniformly Ultimately Bounded (UUB)**
- Lyapunov bound:

$$V(t) \leq \frac{\beta}{\alpha}$$

Tracking error bound:

$$\parallel e(t) \parallel \leq \epsilon, \epsilon = \sqrt{\frac{\beta}{\alpha\lambda_{min}(P)}}$$

$$\epsilon = 3.183319e\text{-}01\ rad$$



Derivation (Control and Stability)

Fuel-Depleted Mode (Non-Adaptive)
Lyapunov function:

$$V_2 = e^T P e + z^2$$

Derivative:

$$\dot{V}_2 = -e^T Q e + 2e^T P d(t) + 2z e_1 - 0.6z^2$$

Bounding:

$$\dot{V}_2 \leq -\alpha_e \parallel e \parallel^2 - \beta_z z^2 + \delta$$

Where:

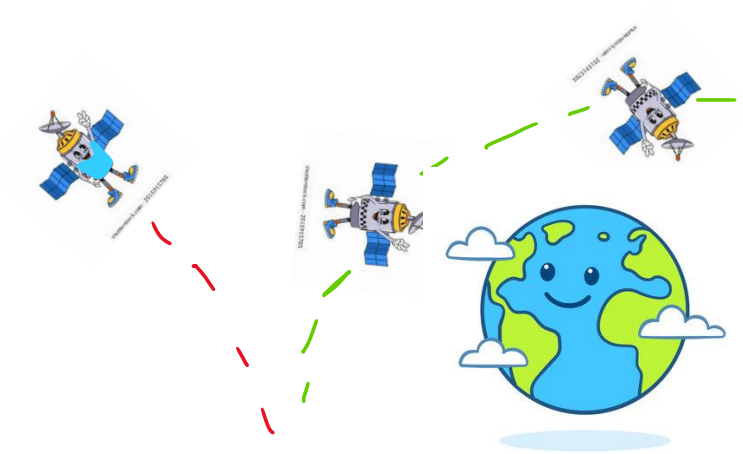
$$\alpha_e = \frac{\lambda_{min}(Q)}{2} - 1$$

$$\beta_z = 0.6$$

$$\delta = \frac{2 \parallel P \parallel^2 d_{max}^2}{\lambda_{min}(Q)}$$

$$\delta = 1.053098\text{e-}09$$

Conclusion:
•Fuel-depleted mode is also **UUB stable**



Simulation

🚀 **What the Model Does**

✅ Nadir pointing tracking
Tracks Earth-pointing attitude using orbital motion reference $\theta_d = \omega_{orbit} t + \pi$

✅ Circular orbit reference
Orbit generated from $\omega_{orbit} = \sqrt{\mu / r^3}$ with kinematic propagation

✅ MRAC adaptive control (σ-modification)
Adaptive state-feedback control with parameter update law active only in coupled mode

✅ Gravity-gradient disturbance
External torque τ_{gg} injected into plant dynamics

✅ Reaction wheel + thruster allocation
Control input split under saturation constraints between reaction wheels and thrusters

✅ Time-varying slosh + mass dynamics
Fuel-dependent inertia, damping, and stiffness variation through $\alpha(t)$

✅ Fuel depletion (hybrid switching system)
Switches between coupled adaptive mode and rigid non-adaptive mode based on fuel fraction

✅ Integral action with leakage
Integral state used for steady-state error reduction with leakage term for stability

✅ Reference model tracking
Two-mode reference model (coupled / rigid) driven by nadir pointing command

✅ Multi-controller comparison
Simulation includes open-loop, PID, and MRAC for performance comparison

✅ Stability and UUB verification
Lyapunov-based analysis confirming uniformly ultimately bounded tracking behavior



Simulation

```
[t,x] = ode45(@(t,x) f(t,x),
  tspan, x0);
```

The code is using Euler Integration because:

✓ Pros

- simple
- fast
- easy to embed control/adaptation loops

⚠️ Cons

- numerical error accumulates
- stability depends heavily on `dt = 0.05`
- can distort fast dynamics (especially slosh + adaptive loop)

A more robust alternative would be:

- `ode45` for plant
- or fixed-step RK4 (best compromise for control simulations)

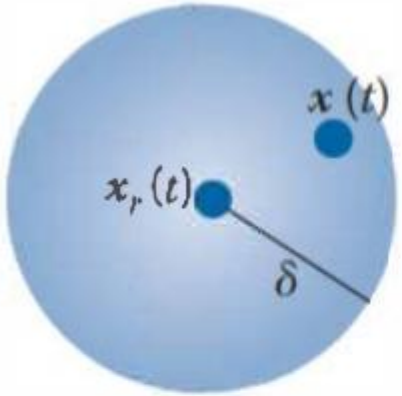
```
%% ========================
% CONTROL GAINS
% ========================
K_coupled = [6 8 4 3];   % (feedback gains → N·m per state unit)
K_rigid   = [10 12 6 5]; % same units

kr_coupled = 1.0;        % (feedforward gain → N·m/rad)
kr_rigid   = 0.7;

Gamma = 2 * eye(4);    % adaptation gain (1/s)
sigma = 0.01;            % leakage (1/s)
Ki = 0;                % integral gain (N·m per state)

u_max_rw  = 0.5;         % N·m (reaction wheel limit)
u_max_thr = 2.0;         % N·m (thruster limit)

W_max = 10;              % adaptive weight limit (N·m scaling)
```



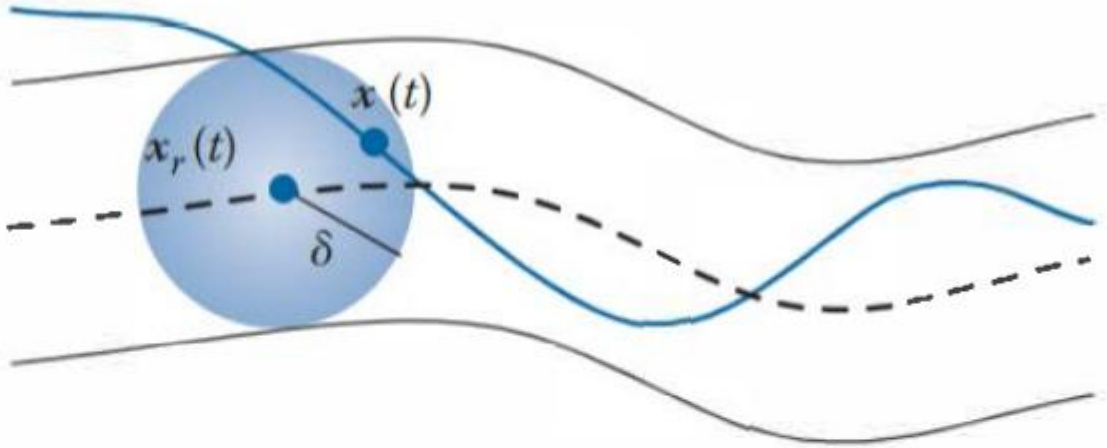
```
%% ========================
% PID GAINS
% ========================
Kp = 10;          % N·m/rad
Ki_pid = 0.2;     % N·m/(rad·s)
Kd = 4;           % N·m·s/rad
```

Results

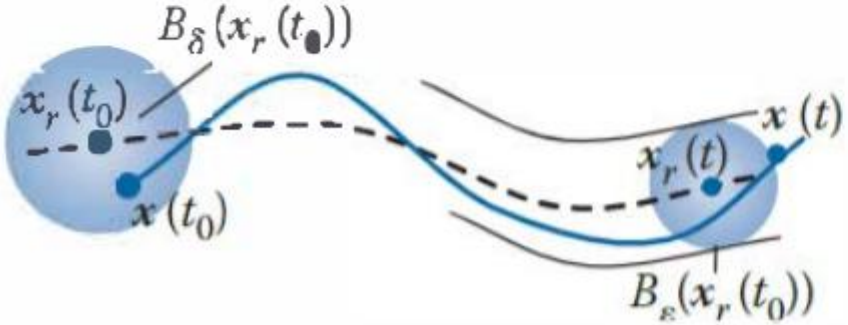
a) Euclidean neighborhood $B_\delta(x_r)$



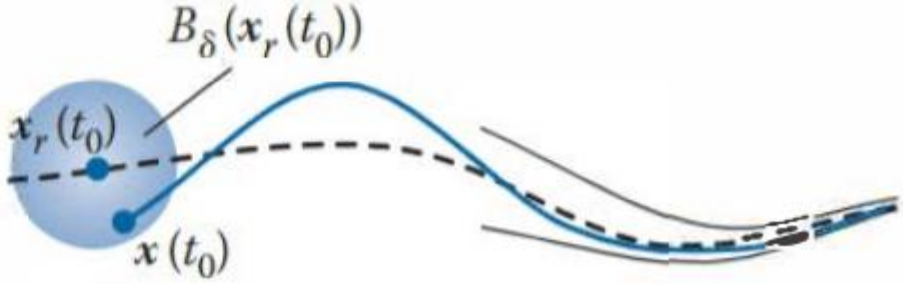
b) Lagrange stability where δ is independent of initial conditions



c) Lyapunov stability where the final convergence domain size ε is a function of the initial conditions

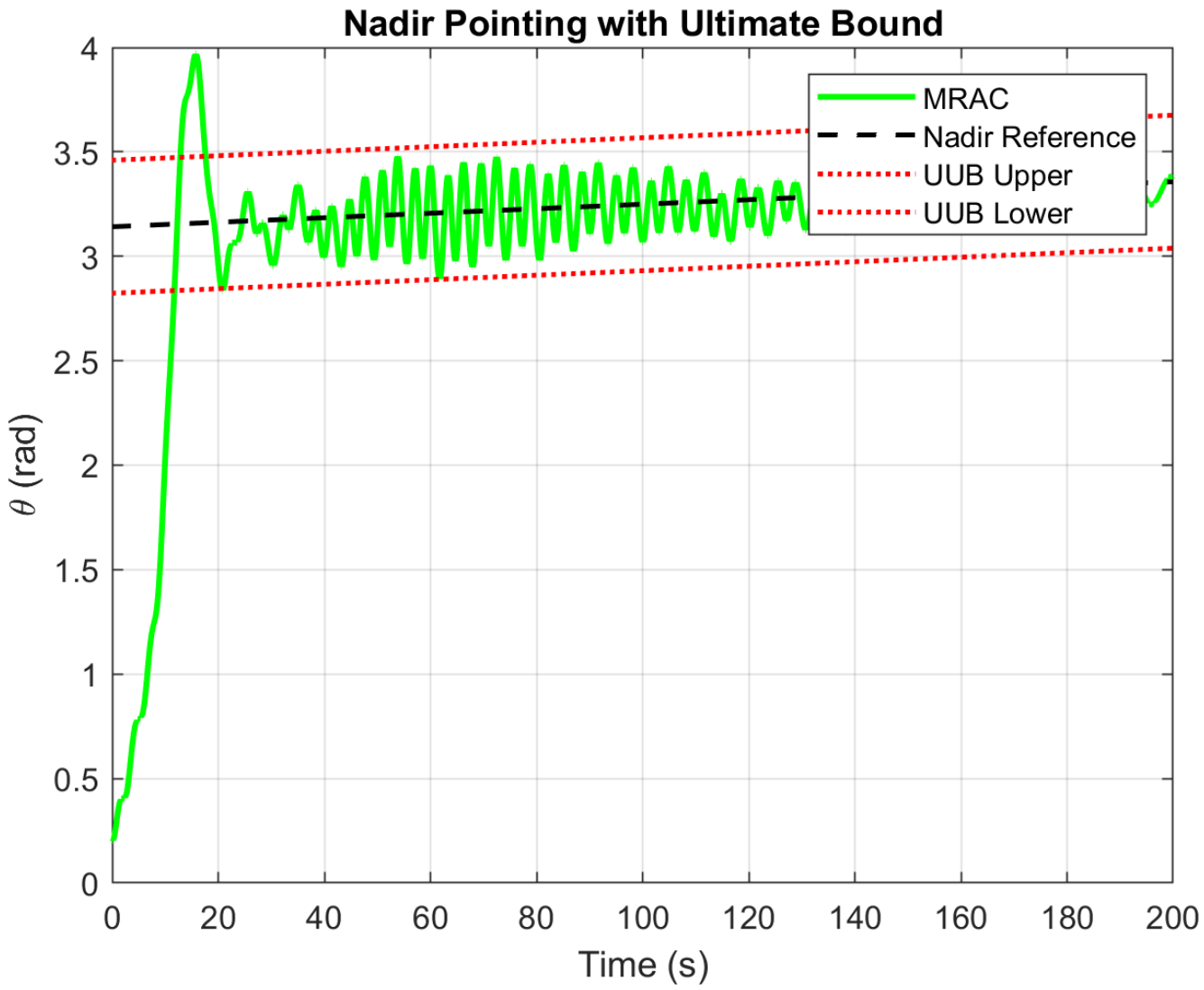


d) Asymptotic Lyapunov stability where the final convergence domain is zero

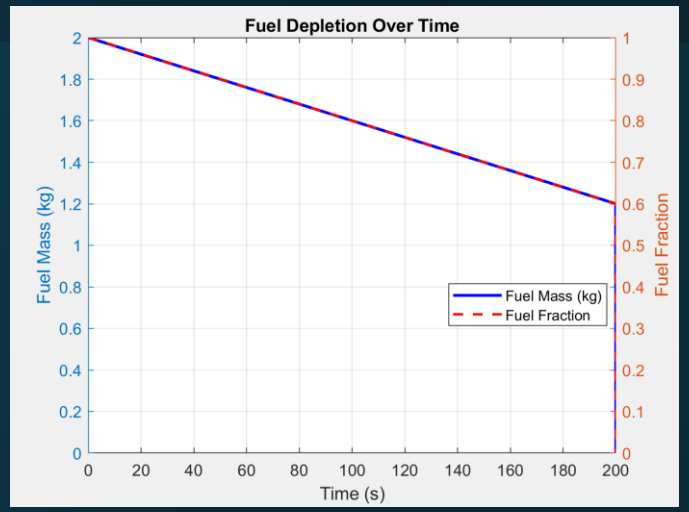


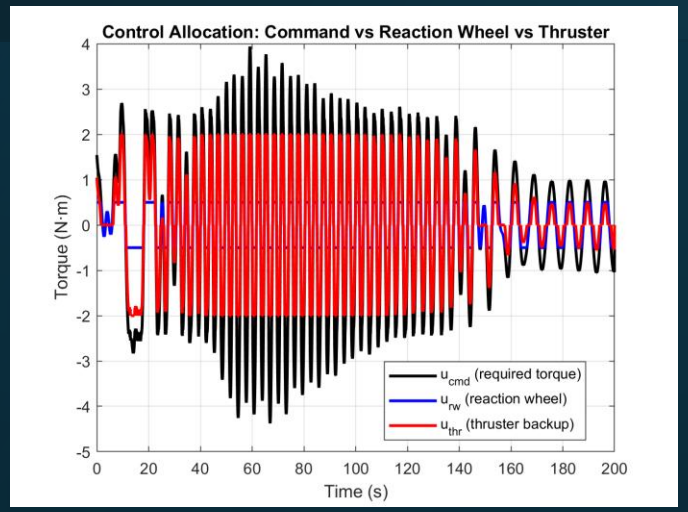
[4] Peng, H., AE544 Analytical Dynamics Lecture Notes, Embry-Riddle Aeronautical University, Daytona Beach, FL, Spring 2026.

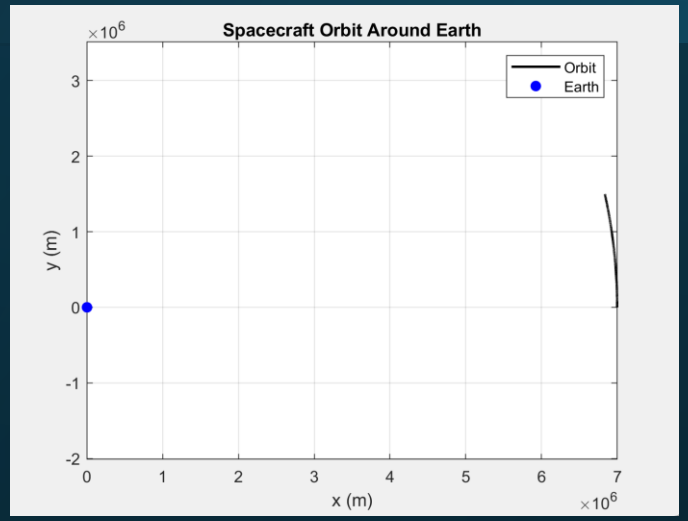
Results


Nadir Pointing with Ultimate Bound
MRAC
Nadir Reference
UUB Upper
UUB Lower
θ (rad)
Time (s)
0
0.5
1
1.5
2
2.5
3
3.5
4
0
20
40
60
80
100
120
140
160
180
200

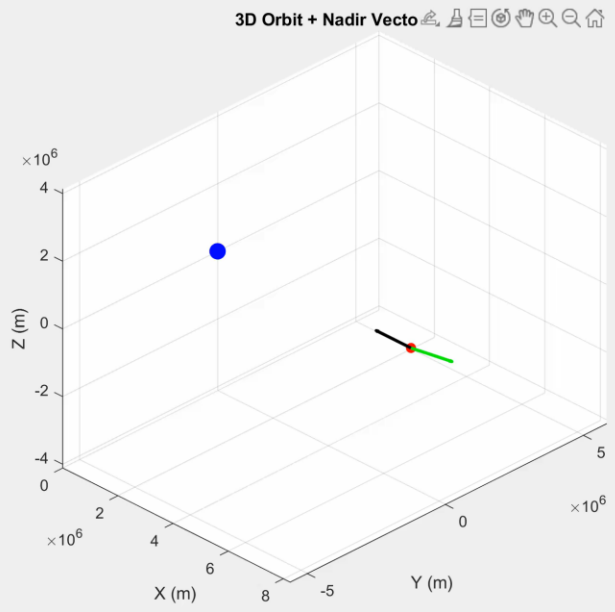
Results

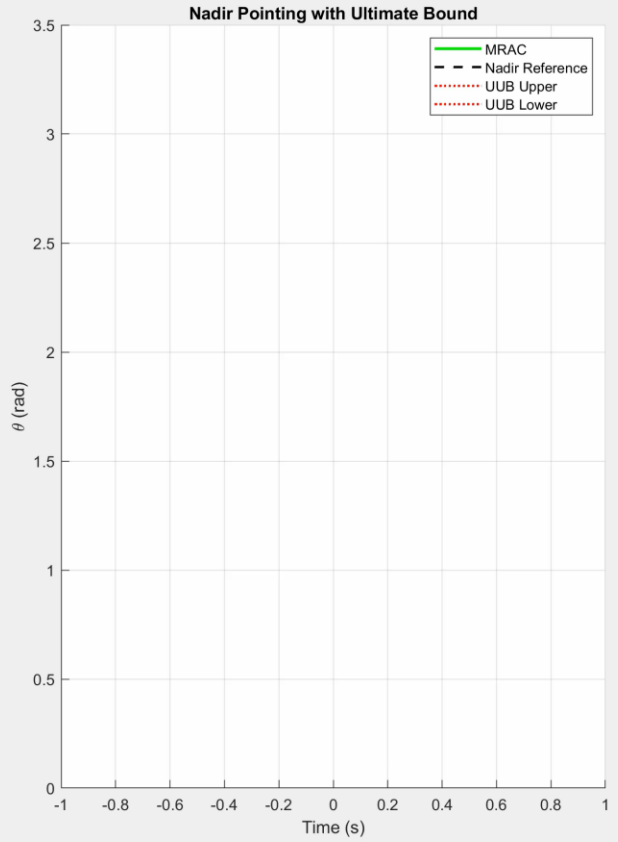

Fuel Depletion Over Time
Fuel Mass (kg)
Fuel Fraction
Time (s)
Fuel Mass (kg)
Fuel Fraction


Control Allocation: Command vs Reaction Wheel vs Thruster
Torque (N·m)
Time (s)
u_cmd (required torque)
u_rw (reaction wheel)
u_thr (thruster backup)


Spacecraft Orbit Around Earth
×10^6
y (m)
x (m)
×10^6
Orbit
Earth

Results


3D Orbit + Nadir Vecto
×10^6
4
2
0
-2
-4
Z (m)
0
2
4
6
8
X (m)
×10^6
-5
0
5
Y (m)
×10^6


Nadir Pointing with Ultimate Bound
MRAC
Nadir Reference
UUB Upper
UUB Lower
θ (rad)
3.5
3
2.5
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Time (s)

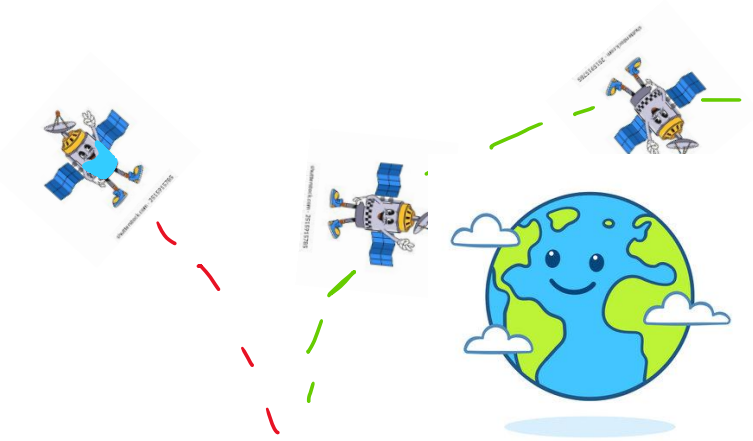
Further Digging: Robust control or PID?

👉 **PID can beat the Robust controller when:**

- System is **slowly varying or nearly LTI**
- Disturbances are small
- Have **good gain tuning + anti-windup**
- PID is poorly tuned or overly conservative
- Reacts to small errors aggressively

👉 **A robust controller usually wins when:**

- Strong **parameter variation** (fuel burn + slosh)
- Nonlinear coupling
- Time-varying inertia
- Disturbances + uncertainty
- You can do a hybrid PID + MRAC
- Slow adaptation, since the orbit is slow.



Further Digging: MRAC or PID?

- Slow adaptation, since the orbit is slow. Why?

🧠 Root cause (most important)

1. Your orbit is *very slow*

Your orbital rate is:

$$\omega = \sqrt{\frac{\mu}{r^3}} \approx 0.0011 \text{ rad/s}$$

So over 500 seconds:

$$\theta_{orbit} \approx 0.0011 \times 500 \approx 0.55 \text{ rad}$$

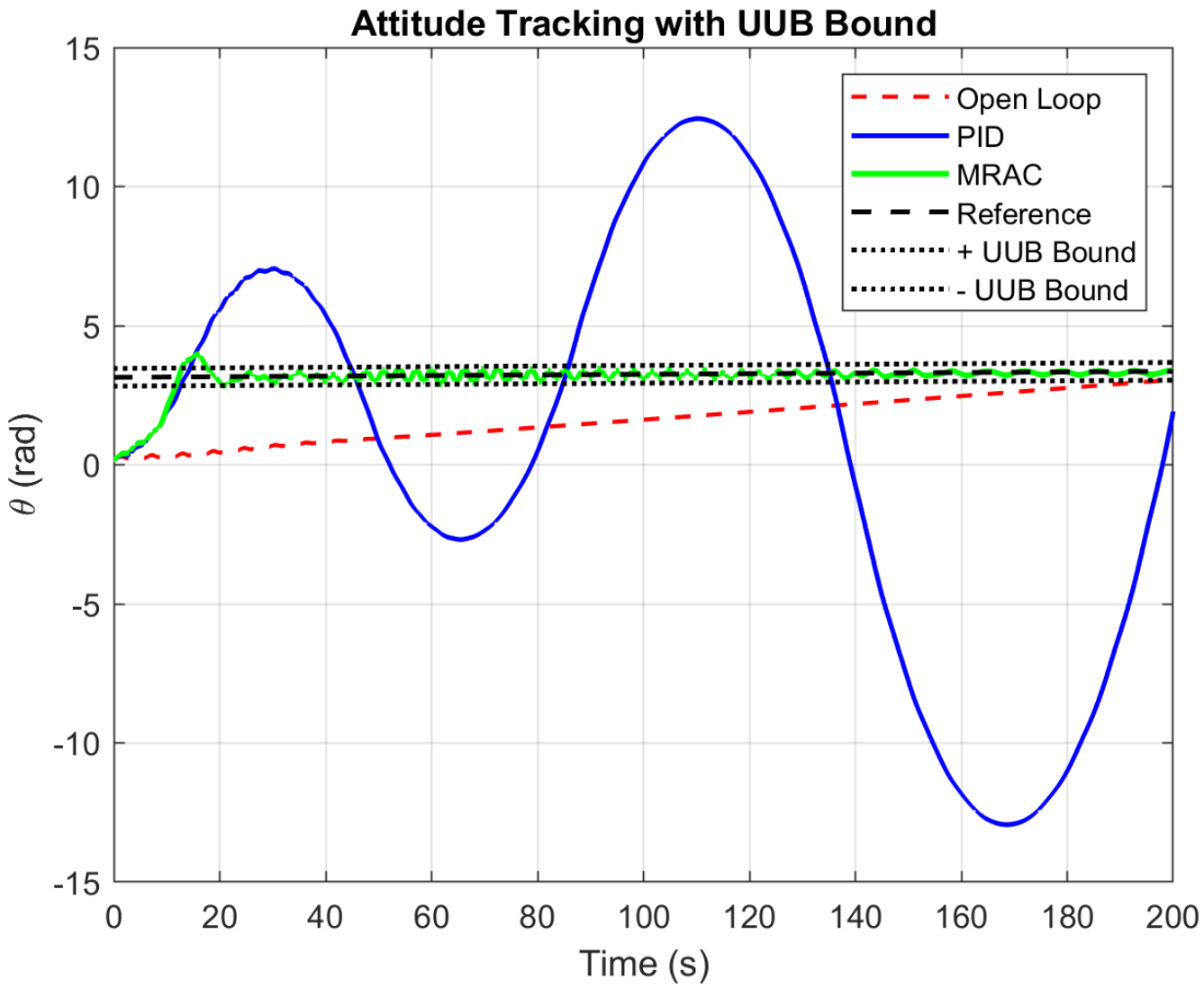
👉 That's only **~31° total motion in 500 seconds**

So the reference:

```
MATLAB
theta_d = theta_orbit + pi;
```

is **barely changing**.

Results


Attitude Tracking with UUB Bound
Open Loop
PID
MRAC
Reference
+ UUB Bound
- UUB Bound
θ (rad)
Time (s)
15
10
5
0
-5
-10
-15
0
20
40
60
80
100
120
140
160
180
200

Analysis

Essential Points

- The controller successfully enables **nadir-pointing orbit tracking** despite nonlinear spacecraft dynamics and slosh effects.
- The Earth orbit is simulated over a limited time window (≈200 s), during which the satellite maintains stable tracking of the reference trajectory.

Actuation and Control Behavior

- The system uses a **dual-actuator architecture**:
  - Reaction wheel for fine control
  - Thruster for high-torque demand compensation
- Reaction wheel saturation occurs under high control effort.
- When saturation is reached, the **thruster compensates to maintain the required torque**, especially under slosh-induced disturbances.

Fuel and System Dynamics

- Fuel depletion is modeled over a longer horizon (≈500 s), while the simulation shown runs for 200 s.
- During the simulation window, sufficient fuel remains, and **slosh dynamics remain active**.
- The robust adaptive controller remains fully operational throughout the simulated period.
- A hybrid robust approach performs better than a rudimentary PID for such multi-state, non-linear, time varying dynamics, without being conservative with the sensor outputs.

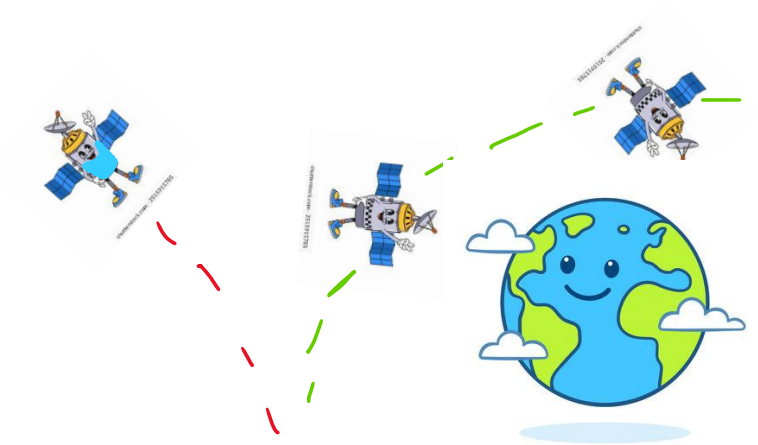
Conclusion

Key Concepts (from this class) Utilized

- Lagrangian Mechanics for system modeling
- Direction Cosine Matrices (DCM) for attitude representation
- Reference Frames and coordinate transformations
- Generalized Coordinates for describing system dynamics
- Equations of Motion (EOM) derivation
- Lyapunov Stability Analysis for stability verification
- Linearization using Jacobian methods
- Linear and Nonlinear System Analysis
- State-Space Representation of dynamic systems
- Orbital Mechanics for spacecraft motion
- Rigid Body Dynamics
- Inertia Tensors and mass distribution effects
- Kinematic Equations for rotational motion

Stability Insight

- The linearized system is Ultimately Uniformly Bounded **UUB**
- Therefore, under standard assumptions, the nonlinear system is at best locally bounded or practically stable



References

[1] Lavretsky, E., and Wise, K. A., Robust and Adaptive Control with Aerospace Applications, Springer-Verlag, 2013.

[2] Vallado, D. A., Analytical Mechanics of Space Systems, 2nd ed., American Institute of Aeronautics and Astronautics, Reston, VA, 2013.

[3] Dogan, M., AE627 Adaptive Control Lecture Notes, Embry-Riddle Aeronautical University, Daytona Beach, FL, Spring 2026.

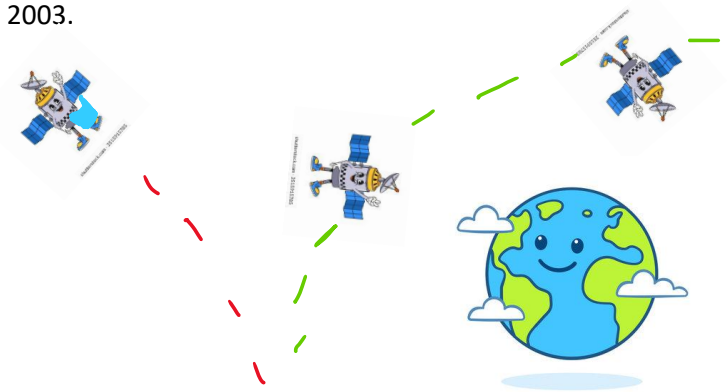
[4] Peng, H., AE544 Analytical Dynamics Lecture Notes, Embry-Riddle Aeronautical University, Daytona Beach, FL, Spring 2026.

[5] Maces-Hernandez, J. A., Seelbinder, D., and Theil, S., "Modeling and Stability Analysis of Sloshing on Liquid-Propelled Reusable Launch Vehicles," DLR eLibrary, German Aerospace Center, available at: https://elib.dlr.de/196124/1/EUCASS2023-549.pdf, accessed 2026.

[6] Chatman, Y., Gangadharan, S., Schlee, K., Ristow, J., Suderman, J., Walker, C., and Hubert, C., "Modeling and Parameter Estimation of Spacecraft Fuel Slosh with Diaphragms Using Pendulum Analogs," NASA Technical Memorandum, NASA/TM-2013-216533, available at: https://ntrs.nasa.gov/api/citations/20130012950/downloads/20130012950.pdf, accessed 2026.

[7] Vallado, D. A., Fundamentals of Astrodynamics and Applications, 4th ed., Microcosm Press, 2013.

[8] Greenwood, D. T., Advanced Dynamics, Cambridge University Press, 2003.



Questions?

